(1) $\mathrm{R}_n(x)=\dfrac{1}{1+x}-\dfrac{1-(-x)^{n+1}}{1-(-x)}$

$=\dfrac{(-x)^{n+1}}{1+x}$

$$\left|\int_0^1 \mathrm{R}_n(x)dx\right| \leqq \int_0^1 |\mathrm{R}_n(x)|dx$$

$$=\int_0^1 \left|\frac{(-x)^{n+1}}{1+x}\right|dx$$

$$<\int_0^1 x^{n+1}dx$$

$$\therefore \quad \left|\int_0^1 \mathrm{R}_n(x)dx\right| < \frac{1}{n+2} \qquad \cdots①$$

同様に、

$$\left|\int_0^1 \mathrm{R}_n(x^2)dx\right| \leqq \int_0^1 \left|\frac{(-1)^{n+1}x^{2n+2}}{1+x^2}\right|dx$$

$$<\int_0^1 x^{2n+2}dx$$

$$\therefore \quad \left|\int_0^1 \mathrm{R}_n(x^2)dx\right| < \frac{1}{2n+3} \qquad \cdots②$$

(2) (ⅰ) $\left|\int_0^1 \mathrm{R}_n(x)dx\right|$

$$=\left|\int_0^1 \frac{1}{1+x}dx-\int_0^1 \{1-x+x^2-\cdots+(-1)^n x^n\}dx\right|$$

$$=\left|\log_e 2-\left\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^n x^{n+1}}{n+1}\right\}\right|$$

$$<\frac{1}{n+2} \to 0 \quad (n\to\infty) \quad (\because \ ①)$$

$$\therefore \quad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log_e 2 \qquad \cdots(答)$$

(ⅱ) $\left|\int_0^1 \mathrm{R}_n(x^2)dx\right|$

$$=\left|\int_0^1 \frac{1}{1+x^2}dx-\int_0^1 \{1-x^2+x^4-\cdots+(-1)^n x^{2n}\}dx\right|$$

$$=\left|\int_0^{\frac{\pi}{4}} d\theta-\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}\right\}\right|$$

$$<\frac{1}{2n+3} \to 0 \quad (n\to\infty) \quad (\because \ ②)$$

$$\therefore \quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4} \qquad \cdots(答)$$