数列 $\{a_n\}$ に対して，$b_n = a_{n+1} - a_n$ で定義される階差数列 $\{b_n\}$ に関して，$\displaystyle\sum_{k=1}^{n} b_k = a_{n+1} - a_1$ が成り立つ.

1. $a_n = [n(n-1)]^2$ としたとき，$b_n = a_{n+1} - a_n$ から，

$$\begin{aligned} b_n &= (n+1)^2 n^2 - n^2(n-1)^2 \\ &= n^2[(n+1)^2 - (n-1)^2] \\ &= n^2[(n^2+2n+1) - (n^2-2n+1)] \\ &= n^2(4n) \\ \therefore\ b_n &= 4n^3 \end{aligned}$$

2. 上の結果および $\displaystyle\sum_{k=1}^{n} b_k = a_{n+1} - a_1$ から，

$$\begin{aligned} \sum_{k=1}^{n} b_k &= a_{n+1} - a_1 = (n+1)^2 n^2 - 0 = 4\sum_{k=1}^{n} k^3 \\ \therefore\ \sum_{k=1}^{n} k^3 &= \frac{1}{4} n^2 (n+1)^2 \end{aligned}$$

3. $a_n = [n(n-1)]^3$ としたとき，$b_n = a_{n+1} - a_n$ から，

$$\begin{aligned} b_n &= (n+1)^3 n^3 - n^3(n-1)^3 \\ &= n^3[(n+1)^3 - (n-1)^3] \\ &= n^3[(n^3+3n^2+3n+1) - (n^3-3n^2+3n-1)] \\ &= n^3(6n^2+2) \\ \therefore\ b_n &= 2n^3(3n^2+1) \end{aligned}$$

4. 上の結果および $\displaystyle\sum_{k=1}^{n} b_k = a_{n+1} - a_1$ から，

$$\begin{aligned} \sum_{k=1}^{n} b_k &= a_{n+1} - a_1 = (n+1)^3 n^3 - 0 = \sum_{k=1}^{n} 2k^3(3k^2+1) \\ 12\sum_{k=1}^{n} k^5 &= 2n^3(n+1)^3 - n^2(n+1)^2 = n^2(n+1)^2[2n(n+1) - 1] \\ \therefore\ \sum_{k=1}^{n} k^5 &= \frac{1}{12} n^2(n+1)^2(2n^2+2n-1) \end{aligned}$$

5. $a_n = [n(n-1)]^4$ としたとき，$b_n = a_{n+1} - a_n$ から，

$$\begin{aligned} b_n &= (n+1)^4 n^4 - n^4(n-1)^4 \\ &= n^4[(n+1)^4 - (n-1)^4] \\ &= n^4[(n^4+4n^3+6n^2+4n+1) - (n^4-4n^3+6n^2-4n+1)] \\ &= n^4(8n^3+8n) \\ \therefore\ b_n &= 8n^5(n^2+1) \end{aligned}$$

6. 上の結果および $\displaystyle\sum_{k=1}^{n} b_k = a_{n+1} - a_1$ から，

$$\sum_{k=1}^{n} b_k = a_{n+1} - a_1 = (n+1)^4 n^4 - 0 = \sum_{k=1}^{n} 8k^5(k^2+1)$$

$$\begin{aligned}
24\sum_{k=1}^{n} k^7 &= 3n^4(n+1)^4 - 2n^2(n+1)^2(2n^2+2n-1) \\
&= n^2(n+1)^2[3n^2(n+1)^2 - 2(2n^2+2n-1)] \\
&= n^2(n+1)^2[3n^2(n^2+2n+1) - 2(2n^2+2n-1)] \\
&= n^2(n+1)^2(3n^4+6n^3+3n^2-4n^2-4n+2) \\
&= n^2(n+1)^2(3n^4+6n^3-n^2-4n+2) \\
\therefore \sum_{k=1}^{n} k^7 &= \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)
\end{aligned}$$

7. $a_n = [n(n-1)]^5$ としたとき，$b_n = a_{n+1} - a_n$ から，

$$\begin{aligned}
b_n &= (n+1)^5 n^5 - n^5(n-1)^5 \\
&= n^5[(n+1)^5 - (n-1)^5] \\
&= n^5[(n^5+5n^4+10n^3+10n^2+5n+1) - (n^5-5n^4+10n^3-10n^2+5n-1)] \\
&= n^5(10n^4+20n^2+2) \\
\therefore b_n &= 2n^5(5n^4+10n^2+1)
\end{aligned}$$

8. 上の結果および $\displaystyle\sum_{k=1}^{n} b_k = a_{n+1} - a_1$ から，

$$\sum_{k=1}^{n} b_k = a_{n+1} - a_1 = (n+1)^5 n^5 - 0 = \sum_{k=1}^{n} 2k^5(5k^4+10k^2+1)$$

$$\begin{aligned}
60\sum_{k=1}^{n} k^9 &= 6n^5(n+1)^5 - 5n^2(n+1)^2(3n^4+6n^3-n^2-4n+2) - n^2(n+1)^2(2n^2+2n-1) \\
&= n^2(n+1)^2[6n^3(n+1)^3 - 5(3n^4+6n^3-n^2-4n+2) - (2n^2+2n-1)] \\
&= n^2(n+1)^2[6n^3(n^3+3n^2+3n+1) - 5(3n^4+6n^3-n^2-4n+2) - (2n^2+2n-1)] \\
&= n^2(n+1)^2(6n^6+18n^5+18n^4+6n^3-15n^4-30n^3+5n^2+20n-10-2n^2-2n+1) \\
&= n^2(n+1)^2(6n^6+18n^5+3n^4-24n^3+3n^2+18n-9) \\
\therefore \sum_{k=1}^{n} k^9 &= \frac{1}{20}n^2(n+1)^2(2n^6+6n^5+n^4-8n^3+n^2+6n-3) \\
&= \frac{1}{20}n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)
\end{aligned}$$