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問題 1

$(x^2+x+1)^{n+1}$ の $x^3$ の係数は、$x^2$ と $x$ をかける場合と、$x$ を 3 回かける場合なので、

$$P_n = \frac{(n+1)!}{1!\times 1!\times (n-1)!} + \frac{(n+1)!}{0!\times 3!\times (n-2)!} = (n+1)n + \frac{(n+1)n(n-1)}{6}$$

$$= (n+1)n\left(1+\frac{n-1}{6}\right) = \frac{1}{6}n(n+1)(n+5)$$

(1)

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}P_k = \lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}\frac{1}{6}k(k+1)(k+5) = \frac{1}{6}\times\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}(k^3+6k^2+5k)$$

$$= \frac{1}{6}\times\lim_{n\to\infty}\frac{1}{n^4}\left[\left\{\frac{1}{2}n(n+1)\right\}^2 + n(n+1)(2n+1) + 5\times\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{6}\times\lim_{n\to\infty}\left[\left\{\frac{1}{2}\times 1\times\left(1+\frac{1}{n}\right)\right\}^2 + \frac{1}{n}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{5}{2\,n^2}\times 1\times\left(1+\frac{1}{n}\right)\right] = \frac{1}{24}$$

(2)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{P_k} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{6}{k(k+1)(k+5)} = 6\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+5)}$$

ここで、

$$\frac{1}{k(k+1)(k+5)} = \frac{1}{5}\left\{\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+5)}\right\}$$

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

$$\sum_{k=1}^{n}\frac{1}{(k+1)(k+5)} = \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+5}\right)$$

$$= \frac{1}{4}\left\{\begin{array}{l} \left(\frac{1}{2}-\frac{1}{6}\right) + \left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{4}-\frac{1}{8}\right) + \left(\frac{1}{5}-\frac{1}{9}\right) \\ +\left(\frac{1}{6}-\frac{1}{10}\right) + \left(\frac{1}{7}-\frac{1}{11}\right) + \left(\frac{1}{8}-\frac{1}{12}\right) + \left(\frac{1}{9}-\frac{1}{13}\right) \\ +\cdots \\ +\left(\frac{1}{n-2}-\frac{1}{n+2}\right) + \left(\frac{1}{n-1}-\frac{1}{n+3}\right) + \left(\frac{1}{n}-\frac{1}{n+4}\right) + \left(\frac{1}{n+1}-\frac{1}{n+5}\right) \end{array}\right\}$$

$$=\frac{1}{4}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}\right)=\frac{1}{4}\left(\frac{77}{60}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}\right)$$

よって、

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{P_k}=\frac{6}{5}\lim_{n\to\infty}\sum_{k=1}^{n}\left\{\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+5)}\right\}$$

$$=\frac{6}{5}\lim_{n\to\infty}\left\{\left(1-\frac{1}{n+1}\right)-\frac{1}{4}\left(\frac{77}{60}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}\right)\right\}=\frac{6}{5}\left(1-\frac{77}{240}\right)=\frac{163}{200}$$

問題 2

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(3n)!}{(2n)!}}=\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{(3n)(3n-1)(3n-2)\cdots(2n+2)(2n+1)}$$

$$=\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+n}{n}\right)\left(\frac{2n+n-1}{n}\right)\left(\frac{2n+n-2}{n}\right)\cdots\left(\frac{2n+2}{n}\right)\left(\frac{2n+1}{n}\right)}$$

$$=\lim_{n\to\infty}\left\{\left(2+\frac{n}{n}\right)\left(2+\frac{n-1}{n}\right)\left(2+\frac{n-2}{n}\right)\cdots\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)\right\}^{\frac{1}{n}}$$

ここで、対数をとります。

$$\lim_{n\to\infty}\log\left\{\left(2+\frac{n}{n}\right)\left(2+\frac{n-1}{n}\right)\left(2+\frac{n-2}{n}\right)\cdots\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)\right\}^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\left\{\log\left(2+\frac{n}{n}\right)+\log\left(2+\frac{n-1}{n}\right)+\log\left(2+\frac{n-2}{n}\right)+\cdots+\log\left(2+\frac{2}{n}\right)+\log\left(2+\frac{1}{n}\right)\right\}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\log\left(2+\frac{k}{n}\right)=\int_0^1\log(x+2)\,dx=\int_0^1\log(x+2)\times(x)'dx$$

$$=[x\log(x+2)]_0^1-\int_0^1\frac{x}{x+2}dx=\log 3-\int_0^1\frac{x+2-2}{x+2}dx=\log 3-\int_0^1\left(1-\frac{2}{x+2}\right)dx$$

$$=\log 3-[x-2\log(x+2)]_0^1=\log 3-\{(1-2\log 3)-(0-2\log 2)\}=3\log 3-2\log 2-1$$

この値を e の指数とします。

$$e^{3\log 3-2\log 2-1}=\frac{\left(e^{\log 3}\right)^3}{(e^{\log 2})^2\times e^1}=\frac{3^3}{2^2e}=\frac{27}{4e}(\cong 2.4831\cdots)$$

追加問題

問題 1

●2 円を次のようにおきます。

原点を中心とする半径 1 の円 $x^2+y^2=1$ … $(B)$

点$D\left(\frac{1}{2},d\right)$を中心とする半径 r の円 $\left(x-\frac{1}{2}\right)^2+(y-d)^2=r^2$ … $(D)$

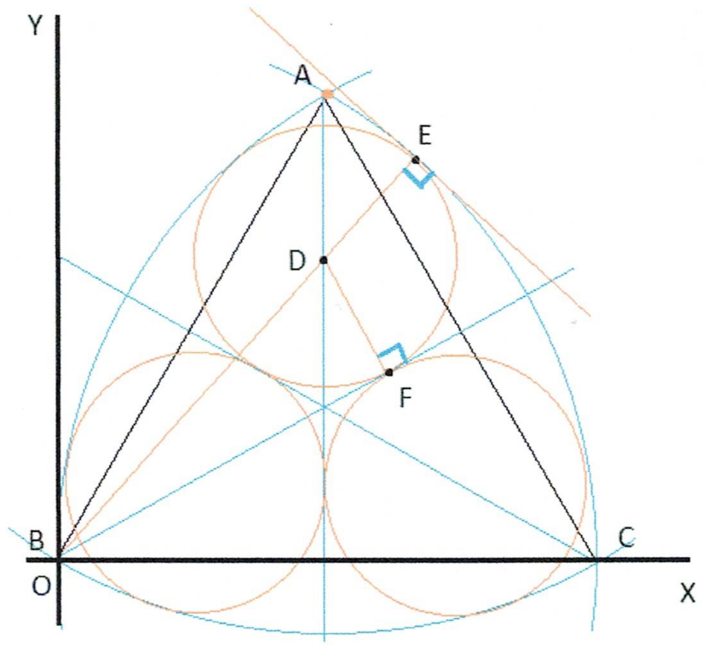
式(B)から式(D)を引くと、

$$
\begin{array}{rl}
 & x^2 \qquad\quad +y^2 \qquad\qquad\quad =1 \\
-) & x^2-x+\frac{1}{4}+y^2-2dy+d^2=r^2 \\
\hline
 & x-\frac{1}{4} \qquad +2dy-d^2=1-r^2
\end{array}
\rightarrow \ -x-2dy+d^2-r^2+\frac{5}{4}=0 \ \cdots \ (E)
$$

式(E)は、円(B)、(D)の点 E での接線です。

点$D\left(\frac{1}{2},d\right)$と接線(E)との距離は r です。

$$r=\frac{-\frac{1}{2}-2d\times d+d^2-r^2+\frac{5}{4}}{\sqrt{(-1)^2+(-2d)^2}} \rightarrow \sqrt{4d^2+1}\times r=\frac{3}{4}-d^2-r^2 \ \cdots \ (1)$$




●円(D)と右下の円の共通接線OFの式は、$y=\frac{1}{\sqrt{3}}x$ → $-x+\sqrt{3}y=0$

点$D\left(\frac{1}{2},d\right)$とこの接線との距離は r です。

$$r=\frac{-\frac{1}{2}+\sqrt{3}d}{\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}} \rightarrow 2r=-\frac{1}{2}+\sqrt{3}d \rightarrow d=\frac{2}{\sqrt{3}}r+\frac{1}{2\sqrt{3}}\cdots \ (2)$$

●式(2)を式(1)に入れると、

$$\sqrt{4d^2+1}\times r=\frac{3}{4}-d^2-r^2 \quad\rightarrow\quad \sqrt{4\left(\frac{2}{\sqrt{3}}r+\frac{1}{2\sqrt{3}}\right)^2+1}\times r=\frac{3}{4}-\left(\frac{2}{\sqrt{3}}r+\frac{1}{2\sqrt{3}}\right)^2-r^2$$

$$\rightarrow\quad \sqrt{\left(\frac{16}{3}r^2+\frac{8}{3}r+\frac{4}{3}\right)r^2}=-\frac{7}{3}r^2-\frac{2}{3}r+\frac{2}{3} \quad\rightarrow\quad \sqrt{(48r^2+24r+12)r^2}=-7r^2-2r+2$$

$$\rightarrow\quad r^4+4r^3-36r^2-8r+4=0 \quad\rightarrow\quad (r^2+8r-2)(r^2-4r-2)=0$$

$$\rightarrow\quad \begin{cases} r^2+8r-2=0 & \rightarrow\quad r=-4\pm 3\sqrt{2} \\ r^2-4r-2=0 & \rightarrow\quad r=2\pm\sqrt{6} \end{cases}$$

この中で正の解は、

$$r=\begin{cases} -4+3\sqrt{2}(=0.2426\cdots) \\ \ 2+\sqrt{6}(=4.4494\cdots) \end{cases}$$

適するのは、$r=-4+3\sqrt{2}(=0.2426\cdots)$

問題 2

このすごろくの上がり方のすべての場合を調べます。

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 回数 | 確率 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | ● | ● | ● | ● | ● | ● | ● | 6 | $1/6^5$ |
| 2 | | | | | | | ● | 5 | $5/6^5$ |
| 3 | | | | | | ● | ● | 5 | $1/6^4$ |
| 4 | | | | | | | ● | 4 | $4/6^4$ |
| 5 | | | | | ● | ● | ● | 5 | $1/6^4$ |
| 6 | | | | | | | ● | 4 | $5/6^4$ |
| 7 | | | | | | ● | ● | 4 | $1/6^3$ |
| 8 | | | | | | | ● | 3 | $3/6^3$ |
| 9 | | | | ● | ● | ● | ● | 5 | $1/6^4$ |
| 10 | | | | | | | ● | 4 | $5/6^4$ |
| 11 | | | | | | ● | ● | 4 | $1/6^3$ |
| 12 | | | | | | | ● | 3 | $4/6^3$ |
| 13 | | | | | ● | ● | ● | 4 | $1/6^3$ |
| 14 | | | | | | | ● | 3 | $5/6^3$ |
| 15 | | | | | | ● | ● | 3 | $1/6^2$ |
| 16 | | | | | | | ● | 2 | $2/6^2$ |
| 17 | | | ● | ● | ● | ● | ● | 5 | $1/6^4$ |
| 18 | | | | | | | ● | 4 | $5/6^4$ |
| 19 | | | | | | ● | ● | 4 | $1/6^3$ |
| 20 | | | | | | | ● | 3 | $4/6^3$ |
| 21 | | | | | ● | ● | ● | 4 | $1/6^3$ |
| 22 | | | | | | | ● | 3 | $5/6^3$ |
| 23 | | | | | | ● | ● | 3 | $1/6^2$ |
| 24 | | | | | | | ● | 2 | $3/6^2$ |
| 25 | | | | ● | ● | ● | ● | 4 | $1/6^3$ |
| 26 | | | | | | | ● | 3 | $5/6^3$ |
| 27 | | | | | | ● | ● | 3 | $1/6^2$ |
| 28 | | | | | | | ● | 2 | $4/6^2$ |
| 29 | | | | | ● | ● | ● | 3 | $1/6^2$ |
| 30 | | | | | | | ● | 2 | $5/6^2$ |
| 31 | | | | | | ● | ● | 2 | 1/6 |
| 32 | | | | | | | ● | 1 | 1/6 |

この表で回数ごとの確率を合計します。

| 回数 | 1 | 2 | 3 | 4 | 5 | 6 | 計 |
|---|---|---|---|---|---|---|---|
| 確率 | 1/6 | $20/6^2$ | $50/6^3$ | $55/6^4$ | $29/6^5$ | $1/6^5$ | 1 |
| $n/6^5$ | 1296 | 4320 | 1800 | 330 | 29 | 1 | 7776 |

よって期待値は、

$$1\times\frac{1296}{7776}+2\times\frac{4320}{7776}+3\times\frac{1800}{7776}+4\times\frac{330}{7776}+5\times\frac{29}{7776}+6\times\frac{1}{7776}$$

$$=\frac{16807}{7776}(\cong 2.1613\cdots)$$