● 問題 440 解答＜三角定規＞

［問題１］(1) 多項定理より

$$(x^2+x+1)^{n+1}=\sum_{(p,q,r)\geqq 0}^{p+q+r=n+1}\frac{(n+1)!}{p!q!r!}(x^2)^p\cdot x^q\cdot 1^r=\sum\frac{(n+1)!}{p!q!r!}x^{2p+q}$$

求めるものは x^3 の係数だから，$P(n)=\dfrac{(n+1)!}{p!q!r!}$ …①，$2p+q=3$ …②，$p+q+r=n+1$ …③

②③より $(p,\ q,\ r)=(0,\ 3,\ n-2),\ (1,\ 1,\ n-1)$ …④

④を①に戻して，

$$P(n)=\frac{(n+1)!}{3!(n-2)!}+\frac{(n+1)!}{(n-1)!}=\frac{(n+1)n(n-1)}{6}+(n+1)n=\frac{1}{6}(n^3+6n^2+n)$$

$$\sum_{k=1}^{n}P(k)=\frac{1}{6}\sum_{k=1}^{n}(n^3+6n^2+5n)$$

$$=\frac{1}{6}\left\{\frac{1}{4}n^2(n+1)^2+n(n+1)(2n+1)+\frac{5}{2}n(n+1)\right\}$$

$$=\frac{1}{24}n(n+1)(n+2)(n+7)$$

以上より，$\displaystyle\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}P(k)=\lim_{n\to\infty}\frac{1}{24}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{7}{n}\right)=\mathbf{\frac{1}{24}}$ …[答]

(2) $$\sum_{k=1}^{n}\frac{1}{P(k)}=\sum_{k=1}^{n}\frac{6}{k(k+1)(k+5)}=\frac{3}{10}\sum_{k=1}^{n}\frac{20}{k(k+1)(k+5)}$$

$$=\frac{3}{10}\sum_{k=1}^{n}\left(\frac{4}{k}-\frac{5}{k+1}+\frac{1}{k+5}\right)=\frac{3}{10}\sum_{k=1}^{n}\left\{4\left(\frac{1}{k}-\frac{1}{k+1}\right)-\left(\frac{1}{k+1}-\frac{1}{k+5}\right)\right\}$$

$$=\frac{3}{10}\left[4\left\{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}\right.$$
$$-\left\{\left(1-\frac{1}{5}\right)+\left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\cdots\right.$$
$$\left.\left.+\left(\frac{1}{n-3}-\frac{1}{n+1}\right)+\left(\frac{1}{n-2}-\frac{1}{n+2}\right)+\left(\frac{1}{n-1}-\frac{1}{n+3}\right)+\left(\frac{1}{n}-\frac{1}{n+4}\right)+\left(\frac{1}{n+1}-\frac{1}{n+5}\right)\ \right\}\ \right]$$

$$=\cdots=\frac{3}{10}\cdot\frac{n(163n^4+2445n^3+13255n^2+30675n+25462}{60(n+1)(n+2)(n+3)(n+4)(n+5)}$$

$$=\frac{163+2445/n+13255/n^2+30675/n^3+25467/n^4}{200(1+1/n)(1+2/n)(1+3/n)(1+4/n)(1+5/n)}$$

$$\overset{n\to\infty}{\to}\ \mathbf{\frac{163}{200}}$$ …[答]

[問題２]

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(3n)!}{(2n)!}} \quad \cdots(\#)$$

$$\log\frac{1}{n}\sqrt[n]{\frac{(3n)!}{(2n)!}}=\frac{1}{n}\sum_{k=1}^{n}\log(2n+k)-\log n$$

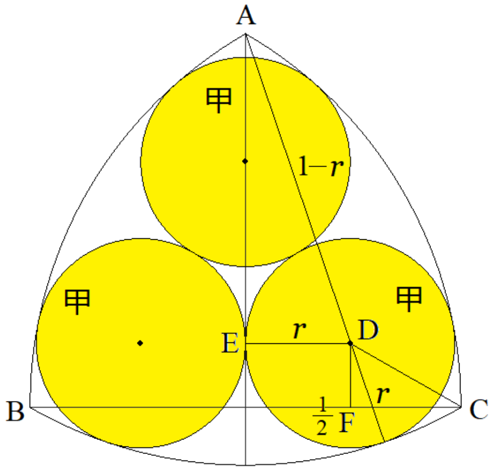
$$=\frac{1}{n}\sum_{k=1}^{n}(\log(2n+k)-\log n)=\frac{1}{n}\sum_{k=1}^{n}\log\left(2+\frac{k}{n}\right)$$

$$\overset{n\to\infty}{\to}\int_0^1\log(2+x)dx=3\log3-3-(2\log2-2)=\log\frac{27}{4}-1=\log\frac{27}{4e}$$

以上より，与式 $(\#)=\mathbf{\frac{27}{4e}}$ $(=2.483\cdots)$ $\cdots$[答]

《追加問題》

[問題１] 右図のように各点を定め，甲円の半径を r とする。



$$\mathrm{DF}=\mathrm{CF}\tan30^\circ=\frac{\sqrt{3}}{3}\left(\frac{1}{2}-r\right)$$

$$\mathrm{AE}=\frac{\sqrt{3}}{2}-\mathrm{DF}=\frac{\sqrt{3}}{3}(1+r),\quad \mathrm{AD}=1-r$$

$\mathrm{AD}^2=\mathrm{AE}^2+\mathrm{DE}^2$：$(1-r)^2=\frac{1}{3}(1+r)^2+r^2$ $\quad\therefore\ r^2+8r-2=0$

$r>0$ でこれを解いて $\quad r=\mathbf{3\sqrt{2}-4}$ $(=0.242\cdots)$ $\cdots$[答]

[問題２]

サイコロを振って出た目の数を左から順に（4, 2, 5, …）のように表記する。表記中例えば「≧ 2」は「2 以上の目が出れば良い」ことを示す。

(ｉ) 1 回で上がりになるのは（6）の場合で，確率 $\frac{1}{6}$ $\cdots$①

(ⅱ) 2 回の場合　出目は (5, ≧1), (4, ≧2), (3, ≧3), (2, ≧4), (1, ≧5)

$$\text{確率}=\frac{1}{6}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{5}{6}+\frac{1}{6}\cdot\frac{4}{6}+\frac{1}{6}\cdot\frac{3}{6}+\frac{1}{6}\cdot\frac{2}{6}=\frac{20}{6^2}\quad\cdots②$$

(ⅲ) 3 回の場合　出目は (4, 1, ≧1), (3, 2, ≧1), (2, 3, ≧1), (1, 4, ≧1)
(3, 1, ≧2), (2, 2, ≧2), (1, 3, ≧2), (2, 1, ≧3), (1, 2, ≧3), (1, 1, ≧4)

$$\text{確率}=\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{6}{6}\cdot4+\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot3+\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{4}{6}\cdot2+\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{3}{6}=\frac{50}{6^3}\quad\cdots③$$

(ⅳ) 4 回の場合　出目は (3, 1, 1, ≧1), (1, 3, 1, ≧1), (1, 1, 3, ≧1), (2, 2, 1, ≧1), (2, 1, 2, ≧1)
(1, 1, 2, ≧1), (2, 1, 1, ≧2), (1, 2, 1, ≧2), (1, 1, 2, ≧2), (1, 1, 1, ≧3)

$$\text{確率}=\frac{1}{6^3}\cdot\frac{6}{6}\cdot6+\frac{1}{6^3}\cdot\frac{5}{6}\cdot3+\frac{1}{6^3}\cdot\frac{4}{6}=\frac{55}{6^4}\quad\cdots④$$

(ⅴ) 5 回の場合　出目は (2, 1, 1, 1, ≧1), (1, 2, 1, 1, ≧1), (1, 1, 2, 1, ≧1), (1, 1, 1, 2, ≧1)
(1, 1, 1, 1, ≧2)

$$確率=\frac{1}{6^4}\cdot\frac{6}{6}\cdot 4+\frac{1}{6^4}\cdot\frac{5}{6}=\frac{29}{6^5}\quad\cdots⑤$$

(ⅵ) 6 回の場合　出目は (1, 1, 1, 1, 1, ≧1)　$確率=\frac{1}{6^5}\cdot\frac{6}{6}=\frac{6}{6^6}$ …⑥

以上より，求める期待値は

$$1\cdot\frac{1}{6}+2\cdot\frac{20}{6^2}+3\cdot\frac{50}{6^3}+4\cdot\frac{55}{6^4}+5\cdot\frac{29}{6^5}+6\cdot\frac{6}{6^6}=\frac{100842}{6^6}=\frac{16807}{6^5}=\mathbf{\frac{7^5}{6^5}}\quad(=2.16\cdots)\quad\cdots[答]$$