４５７．１．$I=\displaystyle\int_0^2\frac{dx}{\sqrt{x^2+1}}$とする．

$x=\tan\theta\left(-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}\right)$とすると，$\dfrac{dx}{d\theta}=\dfrac{1}{\cos^2\theta}$

$x=0$のとき，$\theta=0$

$x=2$のとき，$\theta=\alpha\left(\tan\alpha=2,\ 0<\alpha<\dfrac{\pi}{2}\right)$とする．

$0\leqq\theta\leqq\alpha<\dfrac{\pi}{2}$より，$\cos\theta>0$であるので，

$$\sqrt{x^2+1}=\sqrt{\tan^2\theta+1}=\frac{1}{\cos\theta}$$

$$\therefore I=\int_0^\alpha\cos\theta\cdot\frac{d\theta}{\cos^2\theta}=\int_0^\alpha\frac{(\sin\theta)'}{1-\sin^2\theta}d\theta$$

$t=\sin\theta$とすると，$\dfrac{dt}{d\theta}=\cos\theta$

$$\therefore I=\int_0^{\sin\alpha}\frac{dt}{1-t^2}=-\int_0^{\sin\alpha}\frac{dt}{(t+1)(t-1)}$$

$$=-\frac{1}{2}\int_0^{\sin\alpha}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=-\frac{1}{2}\Big[\log|t-1|-\log(t+1)\Big]_0^{\sin\alpha}$$

$$=\frac{1}{2}\left[\log\frac{t+1}{|t-1|}\right]_0^{\sin\alpha}=\frac{1}{2}\log\frac{1+\sin\alpha}{1-\sin\alpha}$$

ここで，$\tan\alpha=2\left(0<\alpha<\dfrac{\pi}{2}\right)$より，

$$\cos\alpha=\frac{1}{\sqrt{1+\tan^2\alpha}}=\frac{1}{\sqrt{1+2^2}}=\frac{1}{\sqrt{5}}$$

$$\therefore\sin\alpha=\cos\alpha\tan\alpha=\frac{2}{\sqrt{5}}$$

$$\therefore\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{1+\dfrac{2}{\sqrt{5}}}{1-\dfrac{2}{\sqrt{5}}}=\frac{\sqrt{5}+2}{\sqrt{5}-2}=(\sqrt{5}+2)^2$$

$\therefore I=\dfrac{1}{2}\log(\sqrt{5}+2)^2=\log(2+\sqrt{5})$・・・（答）

２． $J=\int_0^{\pi/4}\dfrac{d\theta}{\cos^3\theta}$とすると，

$$J=\int_0^{\pi/4}\frac{\cos\theta}{\cos^4\theta}d\theta=\int_0^{\pi/4}\frac{(\sin\theta)'}{(1-\sin^2\theta)^2}d\theta$$

$t=\sin\theta$とすると，$\dfrac{dt}{d\theta}=\cos\theta$

$$\therefore J=\int_0^{1/\sqrt{2}}\frac{dt}{(1-t^2)^2}=\int_0^{1/\sqrt{2}}\frac{dt}{\{(t+1)(t-1)\}^2}$$

$$=\frac{1}{4}\int_0^{1/\sqrt{2}}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)^2dt$$

$$=\frac{1}{4}\int_0^{1/\sqrt{2}}\left\{\frac{1}{(t-1)^2}-\frac{2}{(t-1)(t+1)}+\frac{1}{(t+1)^2}\right\}dt$$

$$=\frac{1}{4}\int_0^{1/\sqrt{2}}\left[\frac{1}{(t-1)^2}-\left\{\frac{1}{t-1}-\frac{1}{t+1}\right\}+\frac{1}{(t+1)^2}\right]dt$$

ここで，

$$\int_0^{1/\sqrt{2}}\left\{\frac{1}{(t-1)^2}+\frac{1}{(t+1)^2}\right\}dt=\left[-\frac{1}{t-1}-\frac{1}{t+1}\right]_0^{1/\sqrt{2}}$$

$$=-\frac{1}{\frac{1}{\sqrt{2}}-1}-\frac{1}{\frac{1}{\sqrt{2}}+1}=-\frac{\sqrt{2}}{1-\sqrt{2}}-\frac{\sqrt{2}}{1+\sqrt{2}}=-\frac{2\sqrt{2}}{1-2}=2\sqrt{2}$$

$$\int_0^{1/\sqrt{2}}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt=\Big[\log|t-1|-\log(t+1)\Big]_0^{1/\sqrt{2}}$$

$$=\left[\log\frac{|t-1|}{t+1}\right]_0^{1/\sqrt{2}}=\log\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}=\log\frac{\sqrt{2}-1}{\sqrt{2}+1}=\log(\sqrt{2}-1)^2$$

$$=2\log(\sqrt{2}-1)$$

$$\therefore J=\frac{1}{4}\{2\sqrt{2}-2\log(\sqrt{2}-1)\}=\frac{1}{2}\{\sqrt{2}-\log(\sqrt{2}-1)\}$$

$$=\frac{1}{2}\{\sqrt{2}+\log(\sqrt{2}+1)\}$$ ・・・（答）

3．$K=\int_{\alpha}^{\beta}\cos\left(x-\dfrac{\alpha\beta}{x}\right)dx$ （$\alpha\beta>0$）とする．

$t=x-\dfrac{\alpha\beta}{x}$とすると，$\dfrac{dt}{dx}=1+\dfrac{\alpha\beta}{x^2}=\dfrac{x^2+\alpha\beta}{x^2}$

$\alpha\beta>0$，$x^2\geqq 0$より，$x^2+\alpha\beta\neq 0$

$$\therefore\frac{dx}{dt}=\frac{x^2}{x^2+\alpha\beta}=\frac{(x^2+\alpha\beta)-\alpha\beta}{x^2+\alpha\beta}=1-\frac{\alpha\beta}{x^2+\alpha\beta}\cdots①$$

$x=\alpha$のとき，$t=\alpha-\beta$

$x=\beta$のとき，$t=\beta-\alpha$

$t=x-\dfrac{\alpha\beta}{x}$より，$x^2-tx-\alpha\beta=0$

$$\therefore x=\frac{t\pm\sqrt{t^2+4\alpha\beta}}{2}$$

$t\leqq\sqrt{t^2}<\sqrt{t^2+4\alpha\beta}$より，

$$t-\sqrt{t^2+4\alpha\beta}<0<t+\sqrt{t^2+4\alpha\beta}$$

i)．$\alpha>0$，$\beta>0$のとき，$x>0$

$$\therefore x=\frac{t+\sqrt{t^2+4\alpha\beta}}{2}$$

このとき，

$$x^2+\alpha\beta=(tx+\alpha\beta)+\alpha\beta=\frac{1}{2}(t^2+t\sqrt{t^2+4\alpha\beta}+4\alpha\beta)$$

①に代入すると，

$$\begin{aligned}\frac{dx}{dt}&=1-\frac{2\alpha\beta}{t^2+t\sqrt{t^2+4\alpha\beta}+4\alpha\beta}\\&=1-\frac{2\alpha\beta}{\sqrt{t^2+4\alpha\beta}(\sqrt{t^2+4\alpha\beta}+t)}\\&=1-\frac{2\alpha\beta(\sqrt{t^2+4\alpha\beta}-t)}{\sqrt{t^2+4\alpha\beta}\{(t^2+4\alpha\beta)-t^2\}}\\&=1-\frac{\sqrt{t^2+4\alpha\beta}-t}{2\sqrt{t^2+4\alpha\beta}}=1-\frac{1}{2}\left(1-\frac{t}{\sqrt{t^2+4\alpha\beta}}\right)\\&=\frac{1}{2}\left(1+\frac{t}{\sqrt{t^2+4\alpha\beta}}\right)\end{aligned}$$

$$\therefore K=\int_{\alpha}^{\beta}\cos\left(x-\frac{\alpha\beta}{x}\right)dx=\int_{\alpha-\beta}^{\beta-\alpha}\cos t\cdot\frac{1}{2}\left(1+\frac{t}{\sqrt{t^2+4\alpha\beta}}\right)dt$$

$$=\frac{1}{2}\int_{-(\beta-\alpha)}^{\beta-\alpha}\left(\cos t+\frac{t\cos t}{\sqrt{t^2+4\alpha\beta}}\right)dt$$

ここで，$\dfrac{t\cos t}{\sqrt{t^2+4\alpha\beta}}$は奇関数なので，

$$K=\frac{1}{2}\cdot 2\int_0^{\beta-\alpha}\cos t\,dt=\Big[\sin t\Big]_0^{\beta-\alpha}=\sin(\beta-\alpha)$$

ii)．$\alpha<0$，$\beta<0$のとき，$x<0$

$$\therefore x=\frac{t-\sqrt{t^2+4\alpha\beta}}{2}$$

このとき，

$$x^2+\alpha\beta=(tx+\alpha\beta)+\alpha\beta=\frac{1}{2}(t^2-t\sqrt{t^2+4\alpha\beta}+4\alpha\beta)$$

①に代入すると，

$$\frac{dx}{dt}=1-\frac{2\alpha\beta}{t^2-t\sqrt{t^2+4\alpha\beta}+4\alpha\beta}$$

$$=1-\frac{2\alpha\beta}{\sqrt{t^2+4\alpha\beta}(\sqrt{t^2+4\alpha\beta}-t)}$$

$$=1-\frac{2\alpha\beta(\sqrt{t^2+4\alpha\beta}+t)}{\sqrt{t^2+4\alpha\beta}\{(t^2+4\alpha\beta)-t^2\}}$$

$$=1-\frac{\sqrt{t^2+4\alpha\beta}+t}{2\sqrt{t^2+4\alpha\beta}}=1-\frac{1}{2}\left(1+\frac{t}{\sqrt{t^2+4\alpha\beta}}\right)$$

$$=\frac{1}{2}\left(1-\frac{t}{\sqrt{t^2+4\alpha\beta}}\right)$$

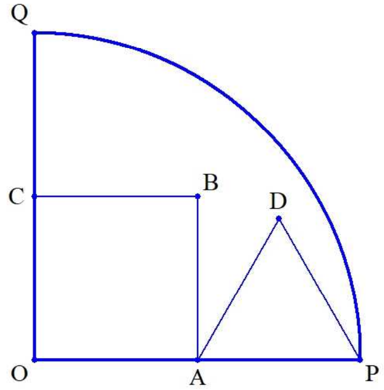
$$\therefore K=\int_\alpha^\beta\cos\left(x-\frac{\alpha\beta}{x}\right)dx=\int_{\alpha-\beta}^{\beta-\alpha}\cos t\cdot\frac{1}{2}\left(1-\frac{t}{\sqrt{t^2+4\alpha\beta}}\right)dt$$

$$=\frac{1}{2}\int_{-(\beta-\alpha)}^{\beta-\alpha}\left(\cos t-\frac{t\cos t}{\sqrt{t^2+4\alpha\beta}}\right)dt$$

$$=\frac{1}{2}\cdot 2\int_0^{\beta-\alpha}\cos t\,dt=\Big[\sin t\Big]_0^{\beta-\alpha}=\sin(\beta-\alpha)$$

以上より，α，βの符号にかかわらず，$K=\sin(\beta-\alpha)$・・・（答）

問題１.

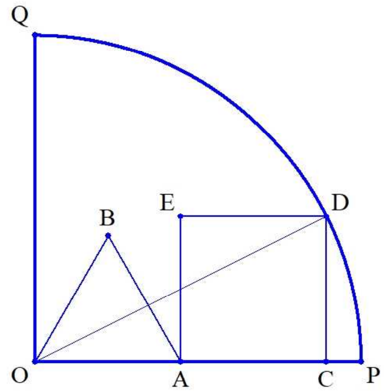


図のように，点O，A，B，C，D、P，Qを定める.

OA＝AP＝rとすると，OP＝OA＋AP＝２ｒ＝１

$\therefore r = \dfrac{1}{2}$・・・（答）

問題２.



図のように，点O，A，B，C，D，E，P，Qを定める.

OA＝AC＝CD＝rとすると，OC＝２ｒ

直角三角形OCDにおいて，$OC^2+CD^2=OD^2$

$\therefore (2r)^2 + r^2 = 1$

$r>0$より，$r = \dfrac{1}{\sqrt{5}}$・・・（答）